3 Reversible processes and cycles. “p dV thermodynamics” for the calculation of thermodynamic engines

3.1 Work and heat for reversible processes

For a heuristically important and qualitatively correct treatment of thermodynamic processes one usually ignores shear stresses, heat conduction, and temperature- and pressure gradients. We have discussed the working $\dot{W}$ of such idealized – reversible – processes in Paragraph 1.5.6. The stress work –or internal work[*] – done in the time $\mathrm{d}t$ is given by

$$\dot{W}_{\text{stress}}\mathrm{d}t = -p\,\mathrm{d}V \ . \tag{3.1}$$

A reversible process is characterized by a pair of time-dependent functions

$$\text{either } T(t),\ V(t)\ ,\ \text{or}\ \ p(t),\ V(t)\ ,\ \text{or}\ \ T(t),\ p(t), \tag{3.2}$$

and it may therefore be represented by a curve in a (T,V)-, or (p,V)-, or (T,p)-diagram, *cf.* Fig. 3.1. If the curve begins at time t_{B} in the point B with p_{B}, V_{B} (say) and ends at time t_{E} in the point E with p_{E}, V_{E}, the transmitted work is obviously

$$W_{\mathrm{BE}} = \int_{t_{\mathrm{B}}}^{t_{\mathrm{E}}} \dot{W}\mathrm{d}t = -\int_{t_{\mathrm{B}}}^{t_{\mathrm{E}}} p\frac{\mathrm{d}V}{\mathrm{d}t}\mathrm{d}t = -\int_{\mathrm{B}}^{\mathrm{E}} p\,\mathrm{d}V\ , \tag{3.3}$$

where the integral is a line integral along the representative curve $p(t)$, $V(t)$ in the (p,V)-diagram. We therefore conclude that the work is graphically represented by the area below the process curve in a (p,V)-diagram as indicated in Fig. 3.1. There is no such easy visualization of the work in the (T,V)- or in the (T,p)-diagram.

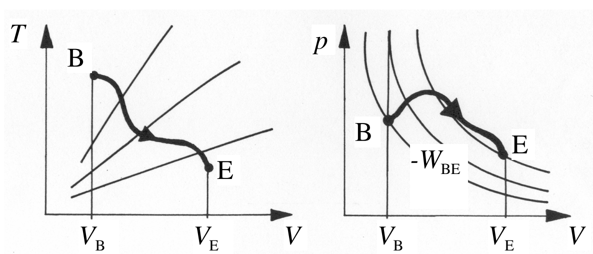

Fig. 3.1 Reversible process in a (T,V)- and a (p,V)-diagram

[*] Both are equal in reversible processes, *cf.* Paragraph 1.5.6.

The transmitted heat cannot be graphically interpreted – at least not in a (p,V,T)-diagram.* But the heat may also be represented by a line integral at least for a closed system of mass m, where we have

$$\dot{Q}\,\mathrm{d}t = \mathrm{d}U + p\,\mathrm{d}V \qquad \text{or} \qquad \dot{Q}\,\mathrm{d}t = \mathrm{d}H - V\,\mathrm{d}p \tag{3.4}$$

$$\dot{Q}\,\mathrm{d}t = \left(\frac{\partial U}{\partial T}\right)_V \mathrm{d}T + \left(\left(\frac{\partial U}{\partial V}\right)_T + p\right)\mathrm{d}V \quad \text{or} \quad \dot{Q}\,\mathrm{d}t = \left(\frac{\partial H}{\partial T}\right)_p \mathrm{d}T + \left(\left(\frac{\partial H}{\partial p}\right)_T - V\right)\mathrm{d}p\,. \tag{3.5}$$

Thus by integration from t_B to t_E we obtain

$$Q_{\mathrm{BE}} = m\left[\int_B^E c_v \mathrm{d}T + \int_B^E \left(\left(\frac{\partial u}{\partial v}\right)_v + p\right)\mathrm{d}v\right] \quad \text{or} \quad Q_{\mathrm{BE}} = m\left[\int_B^E c_p \mathrm{d}T + \int_B^E \left(\left(\frac{\partial h}{\partial p}\right)_T - v\right)\mathrm{d}p\right], \tag{3.6}$$

where c_v and c_p are the specific heats, *cf.* Paragraph 2.3.3.

3.2 Compressor and pneumatic machine. The hot air engine

3.2.1 *Work needed for the operation of a compressor*

The compressor process consists of four steps, *cf.* Fig. 3.2. For simplicity we assume that the pressure behind the moving piston is equal to zero; this may be done without essential loss of generality.

1-2: By the backwards movement of the piston air is drawn into the cylinder by suction through the open entrance valve. We ignore the slight depression of the air caused by the retreating piston and assume that this step occurs at the constant pressure p_{L} of the entrance duct.

2-3: Compression of the air by the forward motion of the piston, while both valves are closed.

3-4: When the pressure reaches the desired value p_{H}, the exit valve opens and the compressed air is pushed into the high-pressure duct at constant pressure.

4-1: When the piston changes direction, the exit valve closes and the entrance valve opens so that the pressure drops from p_{H} to p_{L}, ideally at zero volume.

The pressure during these steps is shown as a function of V on the right hand side of Fig. 3.2. Note that along the branches 1-2 and 3-4 the cylinder represents an open system, since the mass changes in time.

According to (3.3) the work done to the piston reads:

$$W_{12} = -p_{\mathrm{L}} V_2\,,\quad W_{23} = -\int_2^3 p\,\mathrm{d}V\,,\ W_{34} = +p_{\mathrm{H}} V_3\,,\quad W_{41} = 0\,. \tag{3.7}$$

* Later we shall learn about a (temperature, entropy)-diagram in which the heat is represented by the area below the process curve.

Thus obviously the total work is equal to the area within the process curve. The work done along the branch 2-3 depends on the shape of that branch. For the extreme cases of adiabatic and isothermal compression one obtains

$$\begin{array}{ll} \text{adiabatic with} & p_L V_2^{\kappa} = pV^{\kappa} \\ \text{isothermal with} & p_L V_2 = pV \end{array} \qquad W_{23} = \begin{cases} \dfrac{1}{\kappa-1} m \dfrac{R}{M} T_2 \left[\left(\dfrac{p_H}{p_L} \right)^{\frac{\kappa-1}{\kappa}} - 1 \right] \\ m \dfrac{R}{M} T_2 \ln \dfrac{p_H}{p_L} \end{cases} \tag{3.8}$$

Thus the total work for the two cases comes out as

$$\begin{aligned} W^{\text{adiabatic}} &= \frac{\kappa}{\kappa-1} m \frac{R}{M} T_2 \left[\left(\frac{p_H}{p_L} \right)^{\frac{\kappa-1}{\kappa}} - 1 \right] \\ W^{\text{isothermal}} &= m \frac{R}{M} T_2 \ln \frac{p_H}{p_L} . \end{aligned} \tag{3.9}$$

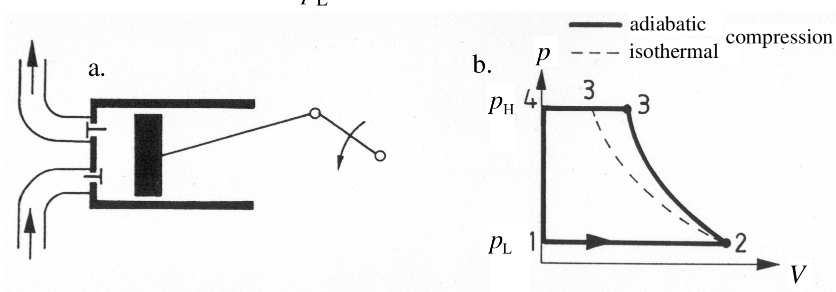

Fig. 3.2 a. Schematic view of a compressor
b. Process curve in a (p,V)-diagram

In the adiabatic case the work required by the compressor is larger than in the isothermal one. This fact is most convincingly confirmed by Fig. 3.2b: Indeed the area on the left of the dashed isothermal line is smaller than the one on the left of the adiabate, since adiabates are steeper than isotherms, *cf.* Paragraph 1.5.8. Actual compressors run so quickly – at several hundred revolutions per minute – that an effective cooling of the air during the compression phase is impossible. One can cool the *wall* of the cylinder. However, this does not produce much of a cooling effect in the interior. Therefore the adiabate of Fig. 3.2 represents the realistic version of the compression although it is undesirable.

It is noteworthy, perhaps, that – by (3.9) – the work required by a compressor depends only on the *ratio* of the pressures p_H and p_L, and *not* on their difference. As a result the compression of air from 1 bar to 10 bar requires just as much work as the compression from 10 bar to 100 bar, although the practical realization of the latter case is more demanding.

3.2.2 *Two-stage compressor*

Now, if we cannot perform an isothermal compression – because of the high number of revolutions – it is still possible to approximate isothermal conditions by building two-stage compressors, *i.e.* two compressors in series. The first one produces an intermediate pressure ratio p_M / p_L adiabatically, and pushes the air at p_M into a cooling duct, where it is cooled back to its initial temperature. The air is then fed into a second compressor, where it reaches the desired pressure p_H, again adiabatically.

Fig. 3.3 shows a (p,V)-diagram of the two-stage process. In this case the work required is, by $(3.9)_1$

$$W = \frac{\kappa}{\kappa-1} m \frac{R}{M} T \left[\left(\frac{p_M}{p_L} \right)^{\frac{\kappa-1}{\kappa}} + \left(\frac{p_H}{p_M} \right)^{\frac{\kappa-1}{\kappa}} - 2 \right]. \tag{3.10}$$

It is smaller than the one-stage process by the shaded area so that there is some profit. This profit is maximal when W as a function of p_M has a maximum and this occurs for

$$p_M = \sqrt{p_H p_L} \,. \tag{3.11}$$

as one can easily check. In this case both compressors require the same work.

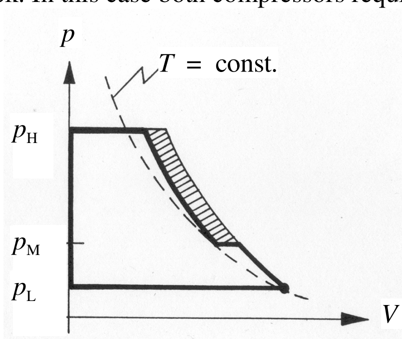

Fig. 3.3 Two-stage compressor

With more than two stages the effective compression curve is a zig-zag line that may approach the desired isotherm in the limit of very many stages. We know of compressors with up to ten stages, and often the cooling ducts are led through the groundwater.

The foregoing discussion about the two-stage compressor is a good example for the heuristic conclusions that may be derived from the consideration of reversible processes.

3.2.3 *Pneumatic machine*

Air of high pressure has many uses. High pressure chisels and high pressure hammers are common applications, *cf.* Fig. 3.4. The process in such a machine is essentially the reverse of the compressor process and in a (p,V)-diagram there are

again four branches. Referring to the figure we may characterize these branches as follows

1-2: High pressure air is drawn into a cylinder.
2-3: Air is decompressed at closed valves.
3-4: Decompressed air is pushed out of the cylinder.
4-1: Change of pressure by closing exit valve and opening high pressure valve.

Because of the speed of the motion of the piston the decompression occurs adiabatically. And, once again, it would be better, if the process ran isothermally, because more work could be gained. In Fig. 3.4 that additional work is represented by the shaded triangular strip.

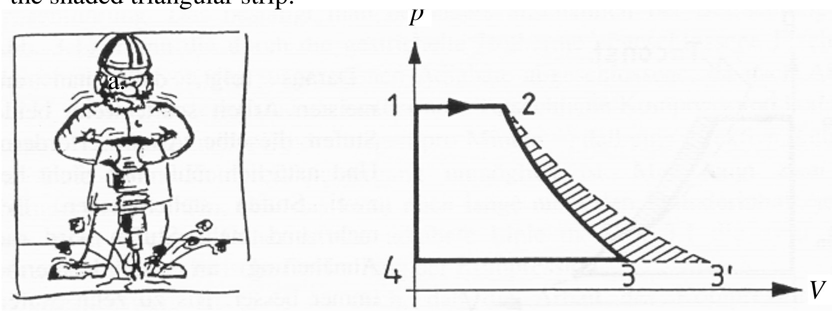

Fig. 3.4 Pneumatic machine

3.2.4 *Hot air engine*

We may consider a situation in which the work of a pneumatic engine is used to drive the compressor which furnishes the compressed air. If both machines ran truly reversibly, such a combination would be possible, albeit as a useless toy that could drive only itself. However, with a small alteration such a coupling of a compressor and a pneumatic machine becomes a useful *heat engine*: The compressed air furnished by the compressor is heated in a heat exchanger. When this is done, the volume increases – at constant pressure – and the adiabatic branch of the process curve of the pneumatic machine is shifted to the right, *cf.* Fig. 3.5. For one revolution of the crank, the work gained is equal to the difference of the areas inside the two process curves.

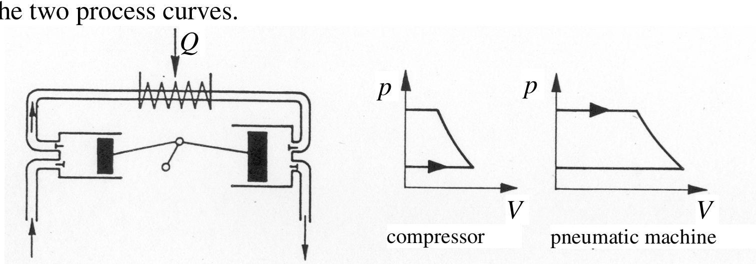

Fig. 3.5 Coupling of a compressor and of a pneumatic machine

This coupled process becomes the prototypical *cycle* of a heat engine when the air exiting the pneumatic machine is cooled and fed back into the compressor, *cf.* Fig. 3.6. In this manner we may construct a hot air engine, the prototype of all heat

engines. The cycle consists of two isobars and two adiabates and it is known as the Joule process.

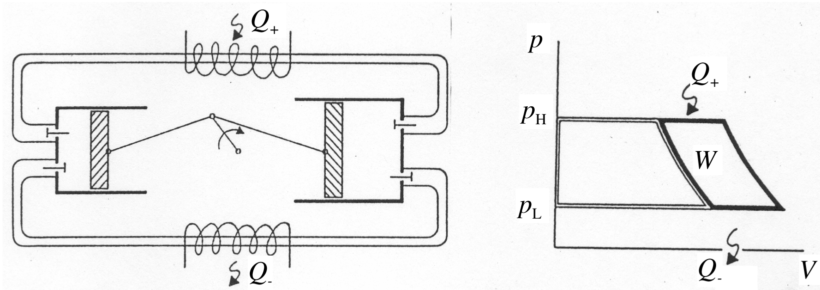

Fig. 3.6 Hot air engine
Q_+ and Q_- denote the heats exchanged in a heater and in a cooler, respectively. The area W represents the work gained.

3.3 Work and heat for reversible processes in ideal gases. "Iso-processes" and adiabatic processes

Many thermodynamic engines perform cycles whose individual branches consist of isotherms, or isochors, or isobars, or which are adiabatic – at least approximately. It is therefore helpful for later reference to make a list of the expressions for work and heat in such "iso-processes." The entries in the list are special cases of the expressions (3.3) and (3.6) for ideal gases, the only case for which we have analytic equations of state.

In an isothermal process the specific work reads

$$w_{\mathrm{BE}} = -\int\limits_{\mathrm{B}}^{\mathrm{E}} p\,\mathrm{d}\upsilon \quad \Rightarrow \quad \text{by } p = \frac{1}{\upsilon}\frac{R}{M}T: \quad w_{\mathrm{BE}} = -\frac{R}{M}T\ln\frac{\upsilon_{\mathrm{E}}}{\upsilon_{\mathrm{B}}}.$$

The specific value q_{BE} of the heat is equal to $-w_{BE}$ by $(3.6)_1$, since u is independent of υ in an ideal gas.

In an isobaric process we have

$$w_{\mathrm{BE}} = -\int\limits_{\mathrm{B}}^{\mathrm{E}} p\,\mathrm{d}\upsilon = -p(\upsilon_{\mathrm{E}} - \upsilon_{\mathrm{B}}) = -\frac{R}{M}(T_{\mathrm{E}} - T_{\mathrm{B}}),$$

and the heat results from $(3.6)_2$ as

$$q_{\mathrm{BE}} = -\int\limits_{\mathrm{B}}^{\mathrm{E}} c_p\,\mathrm{d}T \quad \Rightarrow \quad \text{by } c_p = \frac{\kappa}{\kappa-1}\frac{R}{M}: \quad q_{\mathrm{BE}} = \frac{\kappa}{\kappa-1}\frac{R}{M}(T_{\mathrm{E}} - T_{\mathrm{B}});$$

as always $z = \frac{1}{\kappa-1}$ holds, where κ is the ratio of specific heats.

In an isochoric process the work w_{BE} is obviously zero, while the heat, by $(3.6)_1$, comes out as

$$q_{\mathrm{BE}} = -\int_{\mathrm{B}}^{\mathrm{E}} c_{\upsilon}\,\mathrm{d}T \quad \Rightarrow \quad \text{by } c_{\upsilon} = \frac{1}{\kappa-1}\frac{R}{M}: \quad q_{\mathrm{BE}} = \frac{1}{\kappa-1}\frac{R}{M}\left(T_{\mathrm{E}} - T_{\mathrm{B}}\right).$$

Finally we obviously have $q_{\mathrm{BE}} = 0$ in an adiabatic process and therefore the work is given by

$$w_{\mathrm{BE}} = -\int_{\mathrm{B}}^{\mathrm{E}} p\,\mathrm{d}\upsilon = u(T_{\mathrm{E}}) - u(T_{\mathrm{B}}) \quad \text{hence} \quad w_{\mathrm{BE}} = \frac{1}{\kappa-1}\frac{R}{M}\left(T_{\mathrm{E}} - T_{\mathrm{B}}\right).$$

All these results are summarized in Table 3.1.

Table 3.1 Work and heat for ideal gases in special processes

	isothermal	isobaric	isochoric	adiabatic
work	$-\frac{R}{M}T\ln\frac{\upsilon_{\mathrm{E}}}{\upsilon_{\mathrm{B}}}$	$-\frac{R}{M}\left(T_{\mathrm{E}} - T_{\mathrm{B}}\right)$	0	$\frac{1}{\kappa-1}\frac{R}{M}\left(T_{\mathrm{E}} - T_{\mathrm{B}}\right)$
heat	$\frac{R}{M}T\ln\frac{\upsilon_{\mathrm{E}}}{\upsilon_{\mathrm{B}}}$	$\frac{\kappa}{\kappa-1}\frac{R}{M}\left(T_{\mathrm{E}} - T_{\mathrm{B}}\right)$	$\frac{1}{\kappa-1}\frac{R}{M}\left(T_{\mathrm{E}} - T_{\mathrm{B}}\right)$	0

3.4 Cycles

3.4.1 *Efficiency in the conversion of heat to work*

In a cycle the state of a fixed mass of the working agent changes periodically. In the hot air engine, *cf.* Paragraph 3.2.4, this happened to air and in the steam engine it happens to water. It is then appropriate to apply all thermodynamic calculations to a fixed mass and to write the First Law for closed systems in its specific form as

$$\dot{q}\,\mathrm{d}t = \mathrm{d}u - \dot{w}\,\mathrm{d}t\,. \tag{3.12}$$

Integration over a period – or a complete cycle – gives $\oint \mathrm{d}u = 0$, since the states at the beginning and at the end are equal. Thus (3.12) provides

$$q_{\circ} = -w_{\circ} \quad \text{read: } q_{\text{cycle}} = -w_{\text{cycle}}\,. \tag{3.13}$$

In a heat engine $w_{\circ}$ is negative, since work is gained and, consequently, $q_{\circ}$ is positive. Experience has shown that the total heat $q_{\circ}$ contains positive and negative parts, *i.e.* we have

$$q_{\circ} = q_{+} + q_{-} = q_{+} - \left|q_{-}\right|. \tag{3.14}$$

Therefore we may express (3.13) by saying that the difference between the provided heat and the heat withdrawn is equal to the work.

It is only the provided heat that needs to be paid for in the operation of the heat engine and, of course, it is the work that can be sold. Therefore it makes sense to define the efficiency e of the engine as the ratio of $\left|w_{\circ}\right|$ and q_{+}

$$e = \frac{|w_{\circ}|}{q_+} = 1 - \frac{|q_-|}{q_+}. \tag{3.15}$$

e would be equal to one, if q_- were zero. We shall later see that this is impossible; it contradicts the Second Law, see Chap. 4.

In a *reversible* cycle we have $\dot{w}\,dt = -p\,d\upsilon$ and then the cycle may be represented as a closed loop in a (T,υ)-diagram or a (p,υ)-diagram. In the (p,υ)-diagram the area enclosed by the loop represents the work $w_{\circ}$ of the reversible engine, because the area is equal to $\oint p\,d\upsilon = 0$, *cf.* Fig. 3.7. In a heat engine the states of the working agent move along the closed curve clockwise in the (p,υ)-diagram. And this is usually also the case in a (T,υ)-diagram although there are exceptions: Indeed, if $\left.\frac{\partial p}{\partial T}\right|_{\upsilon} < 0$ holds the states of a heat engine move counter-clockwise along the (T,υ)-curve. This is the case, for instance, for water below 4°C where water behaves "anomalously," see above, Paragraph 2.4.4.

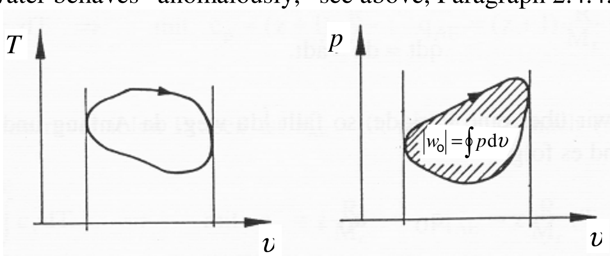

Fig. 3.7 Reversible cycles in a (T,υ)- and a (p,υ)-diagram

3.4.2 *Efficiencies of special cycles*

- *Joule process*

In Paragraph 3.2.4 we have considered the Joule process. That cycle consists of two isobars and two adiabates, *cf.* Fig.s 3.6 and 3.8. The pressures p_1 and p_3 are prescribed.

If the working agent is an ideal gas, the heats and works on the individual branches follow from Table 3.1

$$q_{12} = \frac{\kappa}{\kappa-1}\frac{R}{M}(T_2 - T_1) > 0 \qquad w_{12} = -\frac{R}{M}(T_2 - T_1)$$

$$q_{23} = 0 \qquad w_{23} = \frac{1}{\kappa-1}\frac{R}{M}(T_3 - T_2)$$

$$q_{34} = \frac{\kappa}{\kappa-1}\frac{R}{M}(T_4 - T_3) < 0 \qquad w_{34} = -\frac{R}{M}(T_4 - T_3)$$

$$q_{41} = 0 \qquad w_{41} = \frac{1}{\kappa-1}\frac{R}{M}(T_1 - T_4).$$

$w_{\circ}$ results from summing up all works

$$w_\circ = \frac{\kappa}{\kappa-1}\frac{R}{M}\left[(T_3-T_4)-(T_2-T_1)\right]$$

and, since q_{12} is the only positive contribution to $q_\circ$, we obtain for the efficiency

$$e = 1-\frac{T_3-T_4}{T_2-T_1} = 1-\frac{T_3}{T_2}\frac{1-\frac{T_4}{T_3}}{1-\frac{T_1}{T_2}}\,.$$

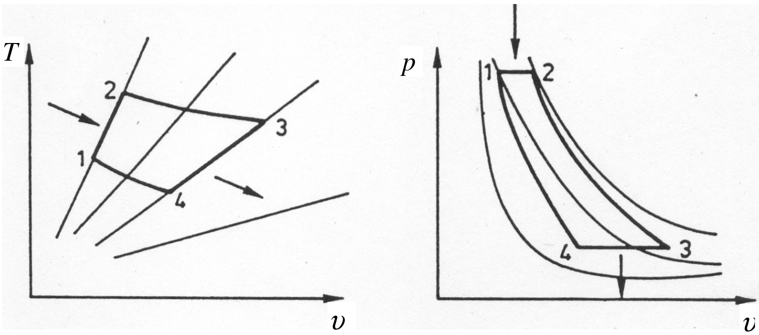

Fig. 3.8 Reversible cycle of the hot air engine

The points 2 and 3 and the points 1 and 4 are connected by the adiabatic equations of state (1.62) and the points 1 and 2, and 3 and 4 lie on isobars. Therefore we have

$$\frac{T_3}{T_2} = \left(\frac{p_3}{p_1}\right)^{\frac{\kappa-1}{\kappa}} \quad \text{and} \quad \frac{T_4}{T_1} = \left(\frac{p_3}{p_1}\right)^{\frac{\kappa-1}{\kappa}} \quad \Rightarrow \quad \frac{T_4}{T_3} = \frac{T_1}{T_2}\,,$$

and the efficiency may be written as

$$e = 1-\frac{T_3}{T_2}\,.$$

This is a correct expression; however, it is not useful, because the corner temperatures T_2 and T_3 are unknown to begin with. What is prescribed, though, are the pressures p_1 and p_3. In terms of these the efficiency reads

$$e = 1-\left(\frac{p_3}{p_1}\right)^{\frac{\kappa-1}{\kappa}}\;; \tag{3.16}$$

it depends on the ratio of the pressures and it is different for different gases, if their κ-values are different. For air with $\kappa = 1.4$ we obtain $e = 0.48$ for a pressure ratio of 10:1.

- *Carnot cycle*

The Carnot engine exchanges heat at two prescribed temperatures only, *i.e.* its cycle consists of two isotherms and two adiabates, *cf.* Fig. 3.9.

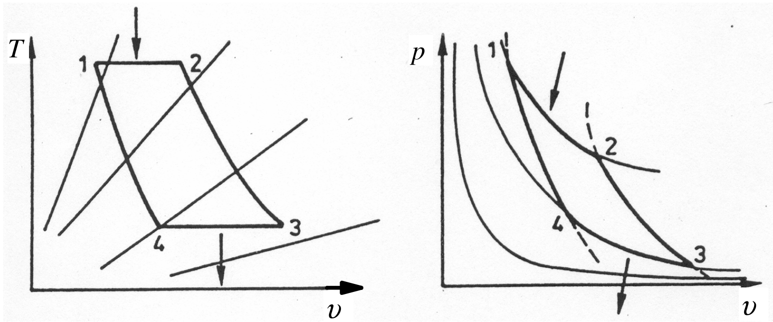

Fig. 3.9 Reversible cycle in a Carnot engine

For an ideal gas heats and works of the branches are given by, *cf.* Table 3.1

$$q_{12} = \frac{R}{M} T_1 \ln\frac{v_2}{v_1} > 0 \qquad\qquad w_{12} = -\frac{R}{M} T_1 \ln\frac{v_2}{v_1}$$

$$q_{23} = 0 \qquad\qquad w_{23} = \frac{1}{\kappa-1}\frac{R}{M}\left(T_3 - T_1\right)$$

$$q_{34} = \frac{R}{M} T_2 \ln\frac{v_4}{v_3} < 0 \qquad\qquad w_{34} = -\frac{R}{M} T_2 \ln\frac{v_4}{v_3}$$

The total work is therefore

$$w_{\circ} - \frac{R}{M} T_1 \ln\frac{v_2}{v_1} - \frac{R}{M} T_3 \ln\frac{v_4}{v_3}$$

and the efficiency reads

$$e = 1 + \frac{T_3}{T_1}\frac{\ln\frac{v_4}{v_3}}{\ln\frac{v_2}{v_1}} \quad \text{or, by} \quad \frac{v_4}{v_3} = \left(\frac{T_1}{T_3}\right)^{\frac{1}{\kappa-1}} \quad \text{and} \quad \frac{v_3}{v_2} = \left(\frac{T_1}{T_3}\right)^{\frac{1}{\kappa-1}}$$

$$e = 1 - \frac{T_3}{T_1}\,. \tag{3.17}$$

The Carnot process and the Carnot efficiency play an important role in the formulation of the Second Law of thermodynamics. Once we have formulated this law we shall see that the Carnot process has a universal efficiency – independent of the working agent – which is maximal among the efficiencies of all cycles with the maximal temperature T_1 and a minimal temperature T_3. Actually, this important result for the Carnot process is forecast by the present result. We can state at this time that the efficiency depends on the prescribed temperature ratio and is independent of the gas. For a given cooling temperature T_3 the Carnot efficiency grows with increasing heating temperature T_1.

- *Modified Carnot cycle*

The heating in reversible processes is always a conceptually difficult phenomenon, because – strictly speaking – it has to occur between a heat reservoir and an

engine (say) of the same temperature. Customary explanations about how this can be realized involve an infinitesimal temperature difference between the reservoir and the engine and infinitely long times for the heating to occur. That is neither practical nor conceptually very satisfactory.

Therefore one may conceive of the heating and cooling phases in a cycle as a realistic problem of heat conduction through auxiliary heat conductors. Thus for instance in a reversible Carnot cycle between T_3 and T_1 , *cf.* Fig 3.9, – with the efficiency

$$e = \frac{|w_\circ|}{q_+} = \frac{q_+ + q_-}{q_+} = 1 - \frac{|q_-|}{q_+} = 1 - \frac{T_3}{T_1}, \tag{3.18}$$

according to (3.17) – the heating may be realized by a conductor with temperatures τ_1 on the hot side and T_1 on the cold side. And the cooling may be realized by a conductor with T_3 on its hot side and τ_3 on the cold side. This means that we have

$$q_+ = C_+(\tau_1 - T_1) \text{ and } q_- = C_-(T_3 - \tau_3), \tag{3.19}$$

where $C_\pm$ are constants depending on the properties of the conductors, *e.g.* the thermal conductivities and the widths of the conductors.

In that case τ_1 and τ_3 are the highest and the lowest temperatures of the engine. Let those be prescribed and let us ask whether the efficiency can be expressed in terms of those temperatures and, if so, what its value is.

Since the efficiency is always equal to

$$e = 1 - \frac{T_3}{T_1} \text{ with } \frac{|q_-|}{q_+} = \frac{T_3}{T_1}, \tag{3.20}$$

we need to express T_1 and T_3 in terms of τ_1 and τ_3. Obviously, by (3.19) and (3.20) we have

$$\frac{T_3}{T_1} = \frac{C_-}{C_+}\frac{T_3 - \tau_3}{\tau_1 - T_1}, \text{ hence } T_3 = \frac{C_- T_1 \tau_3}{(C_+ + C_-)T_1 - C_+\tau_1}. \tag{3.21}$$

The efficiency is therefore equal to

$$e = 1 - \frac{C_-\tau_3}{(C_+ + C_-)T_1 - C_+\tau_1} \tag{3.22}$$

and that is all we can say unless we add another equation in order to determine T_1 as a function of τ_1, τ_3.

We let that additional equation come from the requirement that T_1 be such as to make $|w_\circ|$ maximal. We have

$$\begin{aligned} |w_\circ| &= q_+ - |q_-| && \text{and by (3.19)} \\ &= C_+(\tau_1 - T_1) - C_-(T_3 - \tau_3) && \text{and by } (3.21)_2 \\ &= C_+(\tau_1 - T_1) - C_-\left[\frac{C_- T_1\tau_3}{(C_+ + C_-)T_1 - C_+\tau_1} - \tau_3\right] \end{aligned}$$

which is maximal as a function of T_1 for

$$T_1 = \frac{1}{C_+ + C_-}\left(C_+\tau_1 \pm C_-\sqrt{\tau_1\tau_3}\right). \tag{3.23}$$

This is the required function $T_1 = T_1(\tau_1, \tau_3)$.

Insertion of (3.23) into $(3.21)_2$ gives

$$T_3 = \frac{1}{C_+ + C_-}\left(C_-\tau_3 \pm C_-\sqrt{\tau_1\tau_3}\right) \tag{3.24}$$

and hence follows the efficiency by insertion of (3.23) and (3.24) into $(3.18)_4$ and by some rearrangement

$$e = 1 - \sqrt{\frac{\tau_3}{\tau_1}}. \tag{3.25}$$

The lower sign in (3.23) and (3.24) is irrelevant, since T_1 and T_3 must both be positive.

This is a neat result: The efficiency of the modified Carnot cycle – with realistic heating and cooling – is still given by the ratio of the smallest and highest temperatures. However, it is the *square root* of that ratio that enters the formula. Thus, in a manner of speaking, the efficiency has been decreased by the employment of heat conduction on the isothermal branches of the Carnot process.

- *Ericson cycle*

In the Ericson engine the working agent is carried through a cycle consisting of two isotherms and two isobars, *cf.* Fig. 3.10.

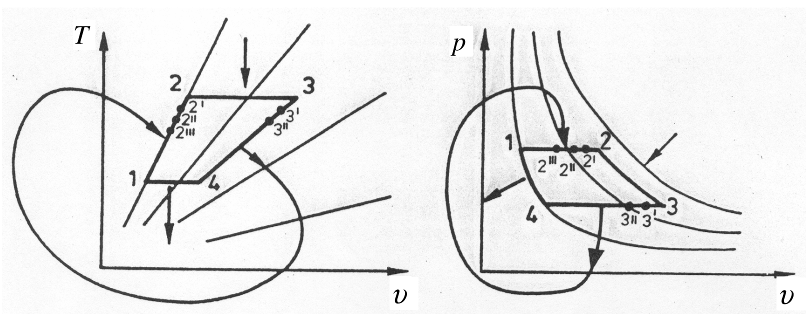

Fig. 3.10 Reversible cycle of an Ericson engine

Works and heats follow from Table 3.1

$$q_{12} = \frac{\kappa}{\kappa-1}\frac{R}{M}\left(T_2 - T_1\right) > 0 \qquad w_{12} = -\frac{R}{M}\left(T_2 - T_1\right)$$

$$q_{23} = \frac{R}{M}T_2 \ln\frac{\upsilon_3}{\upsilon_2} > 0 \qquad w_{23} = -\frac{R}{M}T_2 \ln\frac{\upsilon_3}{\upsilon_2}$$

$$q_{34} = \frac{\kappa}{\kappa-1}\frac{R}{M}\left(T_1 - T_2\right) < 0 \qquad w_{34} = -\frac{R}{M}\left(T_1 - T_2\right)$$

$$q_{41} = \frac{R}{M}T_1 \ln\frac{\upsilon_1}{\upsilon_4} < 0 \qquad w_{41} = -\frac{R}{M}T_1 \ln\frac{\upsilon_1}{\upsilon_4}.$$

The works w_{12} and w_{34} compensate each other and we have $\frac{\upsilon_1}{\upsilon_2} = \frac{\upsilon_4}{\upsilon_3}$, because the points 2 and 3, and 1 and 4 lie on isotherms and because 1 and 2, and 3 and 4 lie on isobars. Therefore the total work is given by

$$w_\circ = \frac{R}{M}(T_2 - T_1)\ln\frac{\upsilon_1}{\upsilon_4}.$$

The heats provided to the process are q_{12} and q_{23}. However, a clever process management will take the heat q_{12} from the process itself, to wit from the heat q_{34} which is equal and opposite to q_{12}, *cf.* Figs. 3.10 and 3.11. If this is done, the only expenditure concerns q_{23} and therefore the efficiency reads

$$e = \frac{|w_\circ|}{q_{23}} = 1 - \frac{T_1}{T_2}, \tag{3.26}$$

so that it is equal to the Carnot efficiency. It is true that in the Ericson process – in contrast to the Carnot process – heats are exchanged on the non-isothermal branches, however, these heats are transferred *within the process*. Such an internal transfer of heat is often called *regeneration*.

Two remarks are appropriate in this context: First of all, an isothermal process is technically unrealistic, because the effective cooling of a fast running engine is impossible. We have discussed this difficulty in Paragraphs 3.2.1 and 3.2.2 and it applies to the Carnot and to the Ericson processes as well. Both represent essentially theoretical possibilities for a conversion of heat into work.

Second, it is all very well to say that the heat q_{34}, which was withdrawn from the process, is supplied to the branch 1-2. Such a shift of heats, however, can only occur from the warmer to a colder body. If the rearrangement of heats is to be realized, it must occur in very small steps: The portion that is released between 3 and $3'$ – *cf.* Fig. 3.10 – must be supplied to the process between $2'$ and $2''$; and what is released between $3'$ and $3''$ must be supplied from $2''$ and $2'''$, *etc.* One way to realize this transfer – at least approximately – is the application of a heat exchanger with flows in opposite directions as indicated in Fig. 3.11.

Compression and decompression are conducted isothermally so that they require cooling and heating, respectively.

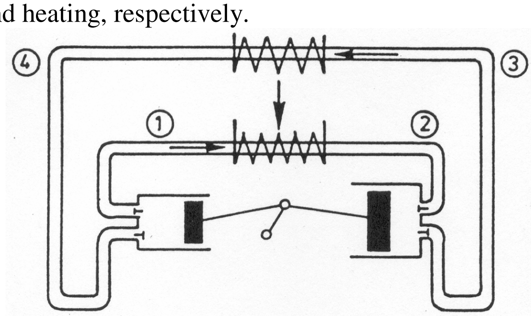

Fig. 3.11 Regeneration in the Ericson cycle

- *Stirling cycle*

In the Stirling engine the cycle consists of two isotherms and two isochors, *cf.* Fig. 3.12.

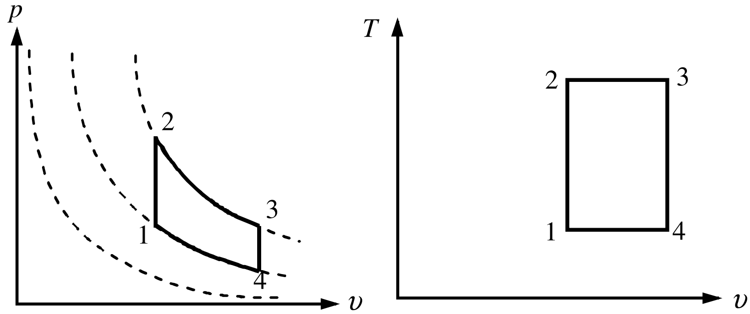

Fig. 3.12 Reversible Stirling cycle

The works and heats on the individual branches are

$$q_{12} = \frac{1}{\kappa - 1}\frac{R}{M}(T_2 - T_1) > 0 \qquad w_{12} = 0$$

$$q_{23} = \frac{R}{M} T_2 \ln\frac{v_3}{v_2} > 0 \qquad w_{23} = -\frac{R}{M} T_2 \ln\frac{v_3}{v_2}$$

$$q_{34} = \frac{1}{\kappa - 1}\frac{R}{M}(T_1 - T_2) < 0 \qquad w_{34} = 0$$

$$q_{41} = \frac{R}{M} T_1 \ln\frac{v_1}{v_4} < 0 \qquad w_{41} = -\frac{R}{M} T_1 \ln\frac{v_1}{v_4} .$$

We face much of the same situation as for the Ericson cycle with the possibility of regeneration: The heating during step 1-2 may be taken from the cooling during step 3-4. And obviously $v_2 = v_1$ and $v_3 = v_4$ holds so that the efficiency reads

$$e = 1 - \frac{T_1}{T_2}, \qquad (3.27)$$

just like for the Carnot process. With an efficient regeneration the Stirling engine run with an ideal gas may thus achieve maximal efficiency.

3.5 Internal combustion cycles

3.5.1 *Otto cycle*

We know that the number of revolutions per minute of the crankshaft of internal combustion engines may reach many thousands, – up to 19,000 –, and that during ignition and during the exhaust process large in-homogeneities occur in the pressure and temperature fields inside the cylinder. And yet we obtain important heuristic results when we treat the process as a reversible one.

Also the working agent is considered as air, although the combustible mixture contains finely dispersed droplets of petrol, and although water vapor and soot particles constitute a good part of the combustion products.

The process in the internal combustion engine is *not* a cycle, because the working agent is regularly exchanged during two out of the four strokes. Indeed, in the Otto engine – or four stroke engine – the following partial processes occur, *cf.* Fig. 3.13 a

0-1: Intake of the combustible mixture (1st stroke).
1-2: Compression of the mixture (2nd stroke).
2-3: Combustion after ignition by a spark plug.
3-4: Expansion (3rd stroke, working stroke).
$4\text{-}1'$: Exhaust after opening of exit valve.
$1'\text{-}0'$: Push-out of the combustion products (4th stroke).

Combustion and exhaust occur so quickly that the motion of the piston may be neglected during these processes; therefore they are considered as isochoric. Compression and expansion are approximately adiabatic processes. The intake and the push-out of the mixture occur at only slightly different pressures. Therefore the corresponding lines are nearly on top of each other so that their contributions cancel each other in the work balance.

With all this in mind one arrives at a substitute process as shown in Fig. 3.13 b. The exhaust process 4-1 is replaced by an isochoric cooling and the intake and push-out parts of the process are missing altogether. The heating along the branch 2-3 is, of course, due to the heat of combustion of the fuel. In this manner the Otto process has been replaced by a fictional cycle of two isochors and two adiabates and this is used for thermodynamic calculations, in particular for the calculation of the efficiency.

The works and heats on the branches of the substitute process are taken from Table 3.1

$$
\begin{aligned}
q_{12} &= 0 & w_{12} &= \tfrac{1}{\kappa-1}\frac{R}{M}\left(T_2 - T_1\right) \\
q_{23} &= \tfrac{1}{\kappa-1}\frac{R}{M}\left(T_3 - T_2\right) > 0 & w_{23} &= 0 \\
q_{34} &= 0 & w_{34} &= \tfrac{1}{\kappa-1}\frac{R}{M}\left(T_4 - T_3\right) \\
q_{41} &= \tfrac{1}{\kappa-1}\frac{R}{M}\left(T_1 - T_4\right) < 0 & w_{41} &= 0\ .
\end{aligned}
$$

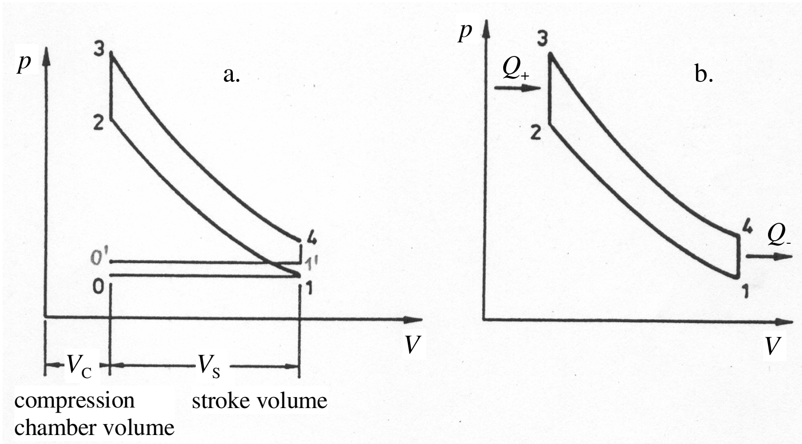

Fig. 3.13 a. 4-stroke process
b. Otto cycle

Therefore the efficiency is given by

$$e = \frac{T_1 - T_2 + T_3 - T_4}{T_3 - T_2} = 1 - \frac{T_4 - T_1}{T_3 - T_2} = 1 - \frac{T_1}{T_2} \frac{1 - \frac{T_4}{T_1}}{1 - \frac{T_3}{T_2}} .$$

Since the points 1 and 2, and the points 3 and 4 are connected by adiabates, we have

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2} \right)^{\kappa - 1} \quad \text{and} \quad \frac{T_3}{T_4} = \left(\frac{V_4}{V_3} \right)^{\kappa - 1} = \left(\frac{V_1}{V_2} \right)^{\kappa - 1} \quad \Rightarrow \quad \frac{T_4}{T_1} = \frac{T_3}{T_2} ,$$

so that the efficiency comes out as

$$e = 1 - \frac{T_1}{T_2} = 1 - \left(\frac{V_2}{V_1} \right)^{\kappa - 1} .$$

Thus the efficiency is determined by the increase of temperature during adiabatic compression. This increase, in turn, is determined by the stroke volume $V_S = V_1 - V_2$ and by the volume $V_C = V_2$ of the compression chamber. If we introduce the compression ratio

$$\varepsilon = \frac{V_1}{V_2} = \frac{V_C + V_S}{V_C} ,$$

we may write the efficiency in the form

$$e = 1 - \frac{1}{\varepsilon^{\kappa - 1}} \tag{3.28}$$

so that it grows with increasing compression ratio.

During the construction of the engine on the drawing board the designer may therefore be tempted to increase ε indefinitely in order to obtain a better efficiency. In practice, however, ε must be smaller than 10, because for a higher

compression – and the concomitant higher temperature – the combustible mixture would ignite prematurely, *i.e.* before the upper dead-center is reached. Obviously this would disturb the proper operation of the engine and, in fact, damage the engine.

Still with $\varepsilon = 8$ (say) and $\kappa = 1.4$ we obtain the efficiency $e = 0.56$. This is not a bad value at all. However, unfortunately, the losses in the actual process through irreversibility and the mechanical losses by friction both reduce the available work by the factor 0.7. Therefore the 4-stroke engine makes use of only about 25% of the energy contained in the fuel. It is difficult to *calculate* these losses reliably, but they may be measured, of course.

3.5.2 *Diesel cycle*

Despite the discrepancy between the calculated and the measured values of the efficiency of the Otto cycle, the formula (3.28) has considerable heuristic value. Indeed, it helps the engineer to realize that he should strive for a higher compression ratio, if he wishes to improve performance. Thus, if a premature ignition of the combustible mixture prevents a higher ratio, – as it does in the Otto cycle the engineer may wish to compress pure air, and add the fuel by injection *after* the compression. This has the additional advantage that the air is so hot after the drastic compression that the fuel – when injected – ignites all by itself, *i.e.* without the help of a spark plug. This new process was invented by Rudolf DIESEL (1858-1913), and it is called the Diesel process; it is a variant of the Otto process and also uses four strokes.

The variation is mostly due to the fact that the injection of the fuel takes time and during that time the piston moves backwards and increases the volume of the cylinder. The pressure increase expected from the combustion of the fuel is reduced by the backward motion of the piston and therefore the injection and combustion occur under nearly isobaric conditions. The substitute process is shown in Fig. 3.14. It has an isobaric branch of length V_I, the injection volume.

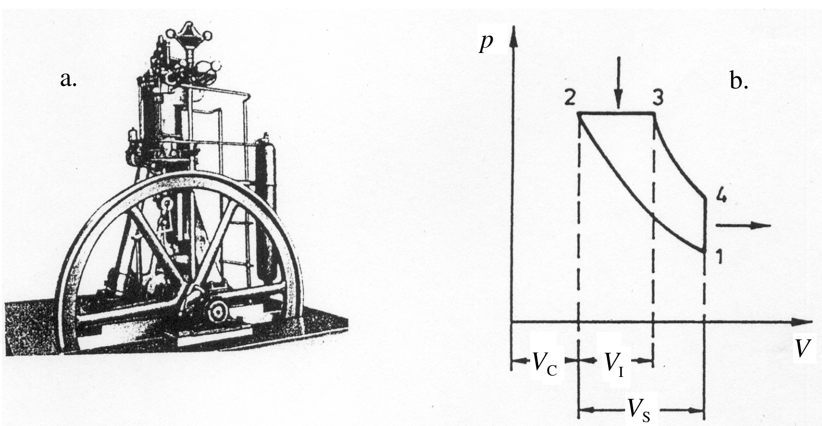

Fig. 3.14 a. First Diesel engine of 1897
b. Substitute cycle for the 4-stroke Diesel process

The heats and works along the different branches may be read off from Table 3.1

$$q_{12}=0 \qquad w_{12}=\tfrac{1}{\kappa-1}\tfrac{R}{M}\left(T_2-T_1\right)$$
$$q_{23}=\tfrac{1}{\kappa-1}\tfrac{R}{M}\left(T_3-T_2\right)>0 \qquad w_{23}=\tfrac{R}{M}\left(T_2-T_3\right)$$
$$q_{34}=0 \qquad w_{34}=\tfrac{1}{\kappa-1}\tfrac{R}{M}\left(T_4-T_3\right)$$
$$q_{41}=\tfrac{1}{\kappa-1}\tfrac{R}{M}\left(T_1-T_4\right)<0 \qquad w_{41}=0\,.$$

Hence follows

$$q_{+}=\tfrac{\kappa}{\kappa-1}\tfrac{R}{M}\left(T_3-T_2\right) \text{ and } w_{+}=-\tfrac{\kappa}{\kappa-1}\tfrac{R}{M}\left(T_3-T_2\right)\left(1-\frac{1}{\kappa}\frac{T_1-T_4}{T_2-T_3}\right)$$

and for the efficiency

$$e=1-\frac{1}{\kappa}\frac{T_1-T_4}{T_2-T_3}=1-\frac{1}{\kappa}\frac{T_1}{T_2}\frac{1-\frac{T_4}{T_1}}{1-\frac{T_3}{T_2}}\,.$$

On the individual branches we have

1-2 (adiabatic): $T_1V_1^{\kappa-1}=T_2V_2^{\kappa-1}\Rightarrow \quad \dfrac{T_1}{T_2}=\left(\dfrac{V_2}{V_1}\right)^{\kappa-1}$

2-3 (isobaric): $\dfrac{T_2}{V_2}=\dfrac{T_3}{V_3} \quad \Rightarrow \quad \dfrac{T_3}{T_2}=\dfrac{V_3}{V_2}$

3-4 (adiabatic): $p_4V_4^{\kappa}=p_3V_3^{\kappa}\Rightarrow \quad \dfrac{p_4}{p_3}=\left(\dfrac{V_3}{V_1}\right)^{\kappa}$

4-1 (isochoric): $\dfrac{T_1}{p_1}=\dfrac{T_4}{p_4}\,.$

Hence follows

$$\frac{T_4}{T_1}=\frac{p_4}{p_1}=\frac{p_3}{p_1}\left(\frac{V_3}{V_1}\right)^{\kappa}=\frac{p_2}{p_1}\left(\frac{V_3}{V_1}\right)^{\kappa}=\left(\frac{V_1}{V_2}\right)^{\kappa}\left(\frac{V_3}{V_1}\right)^{\kappa}=\left(\frac{V_3}{V_2}\right)^{\kappa},$$

and the efficiency becomes

$$e=1-\frac{1}{\kappa}\frac{1}{\left(\frac{V_1}{V_2}\right)^{\kappa-1}}\frac{\left(\frac{V_3}{V_2}\right)^{\kappa}-1}{\frac{V_3}{V_2}-1}\,.$$

We define the compression ratio ε – as before – and in addition the injection ratio φ as

$$\varepsilon=\frac{V_1}{V_2} \quad \text{and} \quad \varphi=\frac{V_3}{V_2}$$

so that

$$e = 1 - \frac{1}{\kappa}\frac{1}{\varepsilon^{\kappa-1}}\frac{\varphi^{\kappa}-1}{\varphi-1}\,. \tag{3.29}$$

For the reasons described above, the Diesel engine tolerates compression ratios that are two to three times larger than those of the Otto engine. If we choose $\varepsilon = 20$ and $\varphi = 2.5$ along with $\kappa = 1.4$, we obtain an efficiency of 0.63. This must be compared with the value 0.56 for the Otto process and we conclude that the Diesel efficiency is higher. In addition the thermodynamic losses reduce the efficiency only by 15%, – rather than 30% for the Otto process –, while the mechanical losses amount to 30% for both. Thus the Diesel process makes use of about 35% of energy contained in the fuel and that is considerably more than the 25% of the Otto engine.

3.5.3 *On the history of the internal combustion engine*

The steam engine, where the heating occurs outside of the working cylinder, preceded the internal combustion engine by more than 100 years. The difficulty was that the fuel had to be a gas or a volatile liquid, and such agents were not available in quantity before petroleum, *i.e.* mineral oil was discovered and exploited in the second half of the 19th century.

The inventor Jean Joseph Etienne LENOIR (1822-1900) used coal gas, *i.e.* the household gas still used in many homes for cooking purposes. This gas is extracted from coal by partial burning. He built the first internal combustion engine and mounted it on a carriage in 1860 thus producing the first "automobile."* LENOIR's engine was quite wasteful of fuel and this made it impractical; so, although Lenoir laid the foundation for an immensely prosperous industry, and although his contribution was recognized, he died a poor man.

Nikolaus August OTTO (1832-1891), a traveling agent, read about LENOIR's engine and improved it by using the four-stroke cycle described above which, in his honor, is often called the Otto process. Otto received a patent for the process in 1877 and founded a firm in Cologne which sold 35,000 engines within a few years.

Otto's assistant Gottlieb Wilhelm DAIMLER (1834-1900) set up a business himself in 1883 and endeavored to build light-weight engines with a good efficiency. He used a *carburetor* which dispersed gasoline into droplets in air and thus produced a combustible mixture. DAIMLER mounted his engine first on a boat, then on a bicycle, and – finally – on a four-wheel car thus producing the first automobile of modern kind. He founded the Daimler motor company in 1890 which produces the Mercedes cars to this very day. They were named after the daughter of DAIMLER's Austrian agent Emil JELLINEK, who had impressed DAIMLER with her beauty and liveliness. Gasoline in those days was bought at drug stores or in pharmacies.

Eventually the automobile became an overwhelming success, as we all know. This was essentially the merit of Henry FORD (1863-1947). In 1908 he invented the "assembly line" for mass production at cheap prizes. His Model T – the Tin Lizzy – was available at US$ 750 and it started modern life as we now know it. Millions of cars with standardized parts were turned out.

Rudolf DIESEL (1858-1913) studied engineering at the Technical High School in Munich. He passed exams with the best grades in Mechanical Engineering ever achieved. Afterwards he worked in the ice factory of the eminent inventor Karl VON LINDE (1842-1934). DIESEL improved the Otto process by fuel injection – doing away with the carburetor and the spark plugs – and his engine had the additional advantage of using cheaper fuel:

* Well, not quite: There had been steam automobiles before.

kerosene instead of gasoline. The first Diesel engine was built in St. Louis in 1897, financed by a brewer of that city. Between World Wars I and II the Diesel engine largely replaced steam engines on ships and locomotives. For a long time the large size and heavy weight of the engine prevented its use in trucks and passenger cars, but nowadays engineers have succeeded to produce lightweight engines that can be used in cars as well. The fuel is cheaper and the consumption is less. However, taxes make these advantages largely irrelevant to the car owner.

DIESEL was much in demand by Navy engineers as a consultant. He was returning from a consultation of the British Navy when he fell from the Channel ferry and drowned. There were wild rumors that the British secret service had a hand in the accident.

3.6 Gas turbine

3.6.1 *Brayton process*

In the Brayton process three essential steps of a heat engine, namely compression, heating and expansion are combined in a single rotating machinery, the gas turbine. The fourth step, cooling, is absent, because the gas turbine uses fresh air from the environment and releases the exhaust gases into the atmosphere, – just like other internal combustion engines.

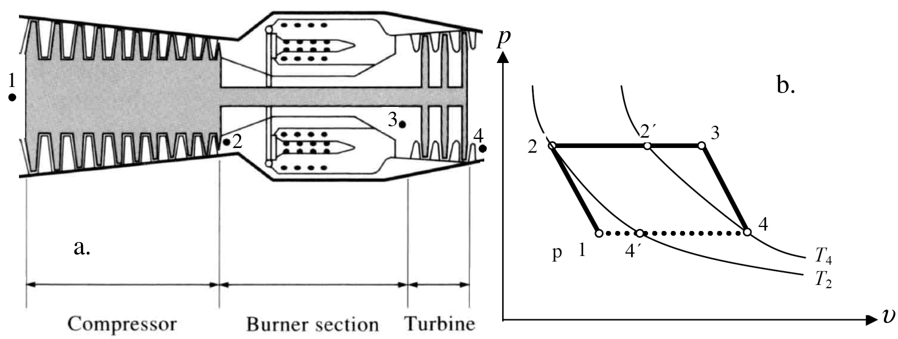

Fig. 3.15 a. Schematic picture of a gas turbine
b. Brayton cycle in a (p, v)-diagram. (The dashed line represents the closure of the substitute cycle)

Fig. 3.15 a shows a schematic picture. The numbers characterize the individual branches as follows.

1-2: adiabatic compression
2-3: isobaric heating by burning the fuel injected into the burner.
3-4: adiabatic expansion in the turbine.

For an efficiency calculation we consider this open process closed into a cycle by replacing the heat loss of the exhaust gases by an isobaric heat exchange that leads back from 4 to 1. The working agent is assumed to be air so that the mass and the properties of the burned fuel are neglected. This is no different from the procedure employed for the internal combustion engines. The resulting substitute process consists of two adiabates and two isobars, *cf.* Fig. 3.15 b, *i.e.* it is a Joule process and has the efficiency (3.16) of the Joule process, *viz.*

$$e = 1 - \frac{T_4}{T_3} = 1 - \left(\frac{p_1}{p_2}\right)^{\frac{\kappa-1}{\kappa}} . \tag{3.30}$$

When a gas turbine is used, the Joule process is known as the Brayton cycle after George BRAYTON, a pioneer of oil burning engines in the 19th century.

There is the possibility for regeneration because the heat withdrawn from the exhaust gases on the branch 4-4´ may be used to preheat the air entering the combustion chamber on the branch 2-2´. In this manner, ideally, it is only the heat $q_{2'3}$ that is really paid for and enters the efficiency. Thus we obtain in a manner often employed before

$$e = \frac{|w_\circ|}{q_{2'3}} = 1 - \frac{T_2}{T_3} = 1 - \frac{T_1}{T_3}\left(\frac{p_2}{p_1}\right)^{\frac{\kappa-1}{\kappa}} . \tag{3.31}$$

Inspection shows that this is significantly larger than the efficiency (3.30) so that regeneration is a useful measure.

3.6.2 *Jet propulsion process*

The most conspicuous use of turbines is for the propulsion of aircraft. In that application the turbine does not serve for the complete expansion – down to atmospheric pressure – of the compressed and heated gas in the burner. Rather the turbine furnishes only the working of the compressor. Thus the gas leaving the turbine is still under high pressure and it expands through a nozzle, – usually supersonically –, in the manner that was discussed in Paragraph 1.5.9. A schematic picture of a turbojet engine is shown in Fig. 3.16 a. Obviously it differs from the gas turbine by the nozzle at the end and also by the conical converging inlet which increases the pressure of the air in front of the turbine. Fig. 3.16 b shows the process in a (p, v)-diagram, and it emphasizes the internal transfer of work from the turbine to the compressor.

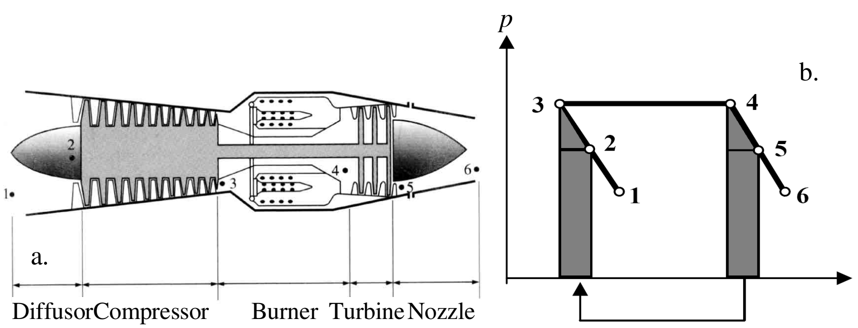

Fig. 3.16 a. Schematic picture of a jet propulsion engine
b. (p, v)-diagram of the jet propulsion process
[adapted from Y.A. Çengel, M.A. Boles, Thermodynamics – An Engineering Approach, 3rd ed., 1998]

If V is the speed of the aircraft and a the speed of the exhaust gas with respect to the craft, the thrust is determined by (1.30) and we have

$$F = \dot{m}(V - a) \, . \tag{3.32}$$

The working, or power, of the thrust is then obviously equal to $\dot{W} = \dot{m}(V - a)V$ so that we may define the *propulsive efficiency* by the ratio

$$\eta = \frac{\dot{W}}{\dot{Q}} = \frac{\dot{m}(V - a)V}{\dot{Q}} \, , \tag{3.33}$$

where $\dot{Q}$ is the heating expended in the burner.

3.6.3 *Turbofan engine*

When jet propulsion was first used in aircraft during WWII, the engines were long and slim, whereas nowadays they appear fat, and even a little stubby. The reason is that the central part of the engine is surrounded by a cylindrical duct, or cowl, or bypass, through which air is propelled by a fan, *cf*. Fig. 3.17. The fan itself is usually driven by a second turbine, a low pressure turbine. In this manner the thrust is increased, because there is a greater accelerated mass $\dot{m}$. Actually there are now two mass rates $\dot{m}_{\text{nozzle}}$ and $\dot{m}_{\text{fan}}$. It is true that the fan does not accelerate the air to supersonic speeds, of course, as the nozzle expansion does, but the bypass ratio $\frac{\dot{m}_{\text{fan}}}{\dot{m}_{\text{nozzle}}}$ may be large, – up to 10 – so that the fan contributes significantly to the overall thrust.

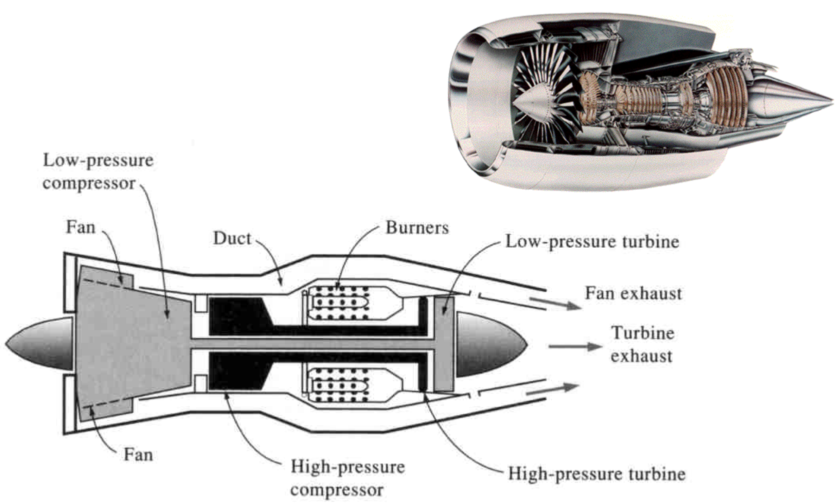

Fig. 3.17 Fanjet engine schematic, and GE GE90-115B high bypass turbofan [adapted from Y.A. Çengel, M.A. Boles, Thermodynamics – An Engineering Approach, 3rd ed., 1998]

Yet another variant of jet propulsion is the *propjet*, where the cowl is eliminated and the fan is replaced by a conventional propeller.